

# Locally accessible information from multipartite ensembles

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We present a universal Holevo-like upper bound on the locally accessible information for arbitrary multipartite ensembles. This bound allows us to analyze the indistinguishability of a set of orthogonal states under LOCC. We also derive the upper bound for the capacity of distributed dense coding with multipartite senders and multipartite receivers.

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It is well known that any set of orthogonal states can be discriminated if there are no restrictions to measurements that one can perform. However, discrimination with certainty is not guaranteed for multipartite orthogonal states, if only local operations and classical communication (LOCC) are allowed [1, 2, 3, 4, 5, 6]. For example, more than two orthogonal Bell states with a single copy cannot be distinguished by LOCC[1]. In Ref. [7] Bennett *et al* constructed a set of orthogonal bipartite pure product states, that cannot be distinguished with certainty by LOCC. Another counterintuitive result was obtained in Ref. [8]: there are ensembles of locally distinguishable orthogonal states, for which one can destroy local distinguishability by reducing the average entanglement of the ensemble states. To understand these interesting results deeply, it is important to investigate the connection between classical and quantum information and extraction of classical information about the ensemble by local operations and classical communication.

An important step is made in Ref.[9], Badzig *et al.* found a universal Holevo-like upper bound on the locally accessible information. They show that for a bipartite ensemble  $\{p_x, \rho_x^{AB}\}$ , the locally accessible information is bounded by

$$I^{LOCC} \leq S(\rho^A) + S(\rho^B) - \max_{Z=A,B} \sum_x p_x S(\rho_x^Z), \quad (1)$$

where  $\rho^A$  and  $\rho^B$  are the reductions of  $\rho^{AB} = \sum_x p_x \rho_x^{AB}$ , and  $\rho_x^Z$  is a reduction of  $\rho_x^{AB}$ .

In this paper, we will prove a multipartite generalization of this bound. First we consider an arbitrary tripartite ensemble  $R = \{p_x, \rho_x^{ABC}\}$  to give an example. The central tool we will require is the following result[9], which is a generalization of the Holevo bound on mutual information.

**Lemma 1.** If a measurement on ensemble  $Q = \{p_x, \rho_x\}$  produces result  $y$  and leaves a postmeasurement ensemble  $Q^y = \{p_{x|y}, \rho_{x|y}\}$  with probability  $p_y$ , then the mutual information  $I$  (between the identity of state in the ensemble and measurement outcome) extracted from the measurement is bounded by

$$I \leq \chi_Q - \bar{\chi}_{Q^y}, \quad (2)$$

where  $\bar{\chi}_{Q^y}$  is the average Holevo bound for the possible postmeasurement ensemble, i.e.,  $\sum_y p_y \chi_{Q^y}$ . Suppose that Alice, Bob and Charlie are far apart and the allowed measurements strategies are limited to LOCC-based measurements. Without loss of generality, let Alice make the first measurement, and suppose that she obtains an outcome  $a$  with probability  $p_a$ . Suppose that the postmeasurement ensemble is  $R_a = \{p_{x|a}, \rho_{x|a}^{ABC}\}$ . Lemma 1 bounds the mutual information obtained from Alice as follows:  $I_1^A \leq \chi_{R^A} - \bar{\chi}_{R_a^A}$ , where  $\chi_{R^A}$  is the Holevo quantity of the  $A$  part of the ensemble  $R$  and  $\chi_{R_a^A}$  is the Holevo quantity of the  $A$  part of the ensemble  $R_a$ . After Bob has learned the Alice's result was  $a$ , his ensemble is denoted by  $R_a^B = \{p_{x|a}, \rho_{x|a}^B\}$ , with  $\rho_{x|a}^B = \text{tr}_{AC}(\rho_{x|a}^{ABC})$ . Suppose Bob performs the second measurement and obtains outcome  $b$  with probability  $p_b$ , then the postmeasurement ensemble is  $R_{ab} = \{p_{x|ab}, \rho_{x|ab}^{ABC}\}$ . Using Lemma 1, the mutual information obtained from Bob's measurement has the bound:  $I_2^B \leq \bar{\chi}_{R_a^B} - \bar{\chi}_{R_{ab}^B}$ , where  $\bar{\chi}_{R_a^B} = \sum_a p_a \left[ S\left(\sum_x p_{x|a} \rho_{x|a}^B\right) - \sum_x p_{x|a} S\left(\rho_{x|a}^B\right) \right]$ , and  $\bar{\chi}_{R_{ab}^B} = \sum_{ab} p_{ab} \left[ S\left(\sum_x p_{x|ab} \rho_{x|ab}^B\right) - \sum_x p_{x|ab} S\left(\rho_{x|ab}^B\right) \right]$ . Similarly, the information extracted from Charlie's measurement is bounded as follows:  $I_3^C \leq \bar{\chi}_{R_{ab}^C} - \bar{\chi}_{R_{abc}^C}$ , where we have assumed that Charlie obtains an outcome  $c$  with probability  $p_c$ . This procedure goes for an arbitrary number of steps, thus the total information gathered from all steps is  $I^{LOCC} = I_1^A + I_2^B + I_3^C + \dots$ , where the subscript  $n$  denotes the information is extracted from the  $n$ th measurement. To proceed with our derivations, we need the following facts:

- (i) Concavity of the von Neumann entropy.
- (ii) A measurement on one subsystem does not change the density matrix at a distant subsystem.
- (iii) A measurement on one subsystem cannot reveal more information about a distant subsystem than about the subsystem itself. For example, after the first measurement by Alice, we have  $\sum_x p_x S(\rho_x^A) - \sum_a p_a \sum_x p_{x|a} S(\rho_{x|a}^A) \geq \sum_x p_x S(\rho_x^B) - \sum_a p_a \sum_x p_{x|a} S(\rho_{x|a}^B)$ .

Suppose that the last measurement is performed by

Alice, then after  $n$  steps of measurements, we obtain the following inequality

$$I^{LOCC} \leq S(\rho^A) + S(\rho^B) + S(\rho^C) - \sum_x p_x S(\rho_x^C) - \sum_{a,b,\dots,n} p_{a,b,\dots,n} S\left(\sum_x p_{x|a,b,\dots,n} \rho_{x|a,b,\dots,n}^A\right), \quad (3)$$

where  $\{p_{x|a,b,\dots,n}, \rho_{x|a,b,\dots,n}^{ABC}\}$  is the postmeasurement ensemble obtained after the measurement in the  $n$ th step and  $p_{a,b,\dots,n}$  is the probability of the sequence of measurement in steps  $1, 2, \dots, n$ . If the last measurement is performed by Bob. We have

$$I^{LOCC} \leq S(\rho^A) + S(\rho^B) + S(\rho^C) - \sum_x p_x S(\rho_x^A) - \sum_{a,b,\dots,(n+1)} p_{a,b,\dots,(n+1)} S\left(\sum_x p_{x|a,b,\dots,(n+1)} \rho_{x|a,b,\dots,(n+1)}^B\right). \quad (4)$$

When the last measurement is performed by Charlie, the inequality takes the form

$$I^{LOCC} \leq S(\rho^A) + S(\rho^B) + S(\rho^C) - \sum_x p_x S(\rho_x^B) - \sum_{a,b,\dots,(n+2)} p_{a,b,\dots,(n+2)} S\left(\sum_x p_{x|a,b,\dots,(n+2)} \rho_{x|a,b,\dots,(n+2)}^C\right). \quad (5)$$

The last terms in Eqs.(3)-(5) are all negative values. Neglecting these terms, we have

$$I^{LOCC} \leq S(\rho^A) + S(\rho^B) + S(\rho^C) - \max_{Z=A,B,C} \sum_x p_x S(\rho_x^Z). \quad (6)$$

For a multipartite ensembles more than three components we can prove the following Lemma by the same way as proving the above results.

**Lemma 2.** For an arbitrary multipartite ensemble  $\{p_x, \rho_x^{B_1 B_2 \dots B_N}\}$ , the maximal locally accessible mutual information satisfies the inequality:

$$I^{LOCC} \leq S(\rho^{B_1}) + S(\rho^{B_2}) + \dots + S(\rho^{B_N}) - \max_{Z=B_1, B_2, \dots, B_N} \sum_x p_x S(\rho_x^Z), \quad (7)$$

where  $\rho_x^{B_n}$  is the reduction of  $\rho^{B_1, B_2, \dots, B_N} = \sum_x p_x \rho_x^{B_1, B_2, \dots, B_N}$  and  $\rho_x^Z$  is a reduction of  $\rho_x^{B_1, B_2, \dots, B_N}$ .

While the ensemble states  $\rho_x^{B_1 B_2 \dots B_N}$  are all pure states, it is possible to write Eq.(7) in a form of the average multipartite q-squashed entanglement. Notice that for the  $N$ -partite pure state  $|\Gamma\rangle_{A_1, \dots, A_N}$ , we have  $[10] E_{sq}^q(|\Gamma\rangle_{A_1, \dots, A_N}) = S(\rho_{A_1}) + \dots + S(\rho_{A_N})$ , where  $\rho_{A_k} = \text{Tr}_{A_1, \dots, A_{k-1}, A_{k+1}, \dots, A_N}(|\Gamma\rangle\langle\Gamma|)$ , then Eq.(7) can be rewritten as  $I^{LOCC} \leq S(\rho^{B_1}) + S(\rho^{B_2}) + \dots + S(\rho^{B_N}) - \sum_x p_x \frac{E_{sq}^q(|\psi\rangle_x^{B_1 B_2 \dots B_N})}{N}$ , where  $|\psi\rangle_x^{B_1 B_2 \dots B_N} \langle\psi| = \rho_x^{B_1, B_2, \dots, B_N}$ . Moreover, noting a recently inequality presented in Ref.[10], for a  $N$ -partite state  $\rho_x^{B_1 B_2 \dots B_N}$ , we have  $\frac{E_{sq}^q(\rho_x^{B_1, B_2, \dots, B_N})}{N} \geq$

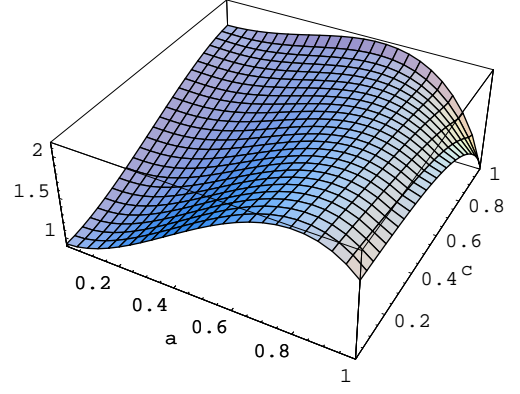


FIG. 1: (Color online). Plot of  $I^{LOCC}$  for the ensemble  $\mathcal{E}_1$ .

$K_D^{(N)}(\rho_x^{B_1, B_2, \dots, B_N})$ , where  $K_D^{(N)}(\rho_x^{B_1, B_2, \dots, B_N})$  denotes the distillable key of the state  $\rho_x^{B_1, B_2, \dots, B_N}$ . Thus Eq.(7) can be further written as  $I^{LOCC} \leq S(\rho^{B_1}) + S(\rho^{B_2}) + \dots + S(\rho^{B_N}) - \sum_x p_x K_D^{(N)}(|\psi\rangle_x^{B_1 B_2 \dots B_N})$ . On the other hand,  $S(\rho^{B_1}) + S(\rho^{B_2}) + \dots + S(\rho^{B_N}) \leq D$ , where  $D = \log_2 d_1 d_2 \dots d_N$ , this gives the following complementarity relation  $I^{LOCC} + \sum_x p_x K_D^{(N)}(|\psi\rangle_x^{B_1 B_2 \dots B_N}) \leq D$ . This inequality shows that the locally accessible information has close relation with the distillable key of the state for the pure ensemble states. We conjecture this relation also holds for the general mixed state ensembles however we were unable to verify or disprove this statement.

**Example 1.** Consider a tripartite ensemble  $\mathcal{E}_1$  consisting (with equal probabilities) of the three states

$$|\psi\rangle_{1,2} = a|000\rangle \pm b|111\rangle, \quad |\psi\rangle_3 = c|001\rangle + d|110\rangle, \quad (8)$$

where we have assumed that  $a, b$  and  $c, d$  are both positive real numbers with  $a(c)^2 + b(d)^2 = 1$ . In Fig.1, we plot the upper bound of  $I^{LOCC}$  for all values of  $a$  and  $c$  with  $0 \leq a(c) \leq 1$  according to Eq.(7).

**Example 2.** Let us evaluate the upper bound of the locally accessible information for the tripartite ensemble  $\mathcal{E}_2$  consisting (with equal probabilities) of the six states

$$\begin{aligned} |\psi\rangle_{1,2} &= a|000\rangle \pm b|111\rangle, \\ |\psi\rangle_{3,4} &= a|001\rangle \pm b|110\rangle, \\ |\psi\rangle_{5,6} &= a|010\rangle \pm b|101\rangle. \end{aligned} \quad (9)$$

Using Lemma 2, we have  $I^{LOCC} \leq -\frac{2}{3}(1+a^2) \log \frac{1}{3}(1+a^2) - \frac{2}{3}(2-a^2) \log \frac{1}{3}(2-a^2)$ , on the other hand, the ensemble  $\mathcal{E}_2$  contains the information  $I = S(\rho^{ABC}) = -a^2 \log \frac{1}{3}a^2 - (1-a^2) \log \frac{1}{3}(1-a^2)$ . For a vivid comparison, we plot  $I^{LOCC}$  and  $I$  in Fig.2. It is shown that  $I^{LOCC} < I$  whenever  $0.222 < a < 0.975$ . Since the locally accessible information extracted is less than the information contained in the ensemble,

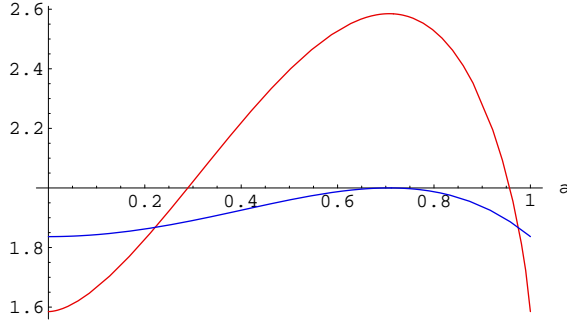


FIG. 2: (Color online). Plots of  $I^{LOCC}$  (blue line) and  $I$  (red line) for the ensemble  $\mathcal{E}_2$ .

it follows immediately that the tripartite ensemble  $\mathcal{E}_2$  consisting of the six states is indistinguishable under LOCC if  $0.222 < a < 0.975$ .

*Example 3.* Consider the following 4-partite ensemble  $\mathcal{E}_3$  consisting (with equal probabilities) of the nine orthogonal states

$$\begin{aligned}
 |\psi\rangle_1 &= \frac{1}{2} (|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle), \\
 |\psi\rangle_2 &= \frac{1}{2} (|0000\rangle - |0011\rangle + |1100\rangle + |1111\rangle), \\
 |\psi\rangle_3 &= \frac{1}{2} (|0001\rangle + |0010\rangle + |1101\rangle - |1110\rangle), \\
 |\psi\rangle_4 &= \frac{1}{2} (|0001\rangle - |0010\rangle + |1101\rangle + |1110\rangle), \\
 |\psi\rangle_5 &= \frac{1}{2} (|0101\rangle + |0110\rangle + |1001\rangle - |1010\rangle), \\
 |\psi\rangle_6 &= \frac{1}{2} (|0101\rangle - |0110\rangle + |1001\rangle + |1010\rangle), \\
 |\psi\rangle_7 &= \frac{1}{2} (|0111\rangle + |0100\rangle + |1011\rangle - |1000\rangle), \\
 |\psi\rangle_8 &= \frac{1}{2} (|0111\rangle - |0100\rangle + |1011\rangle + |1000\rangle), \\
 |\psi\rangle_9 &= \frac{1}{2} (|0000\rangle + |0011\rangle - |1100\rangle + |1111\rangle). \quad (10)
 \end{aligned}$$

In this case, it is easy to show that  $I^{LOCC} \leq 3$ , while the ensemble  $\mathcal{E}_3$  contains the information  $I = \log 9 > I^{LOCC}$ . Thus we conclude that ensemble  $\mathcal{E}_3$  is indistinguishable under LOCC.

As another application of Lemma 2, we can derive an upper bound for the capacity of a scheme of quantum dense coding for multipartite states. Suppose now there are  $N$  Alices, say  $A_1, A_2, \dots, A_N$ , who want to send information to  $M$  receivers, Bobs,  $B_1, B_2, \dots, B_M$ . They share the quantum state  $\rho^{A_1, A_2, \dots, A_N B_1, B_2, \dots, B_M}$ . Using the same techniques as Ref.[11], we can show the capacity of distributed dense coding is bounded by the following quantity:

$$C(\rho) \leq \log_2 d_{A_1} + \dots + \log_2 d_{A_N} + S(\rho^{B_1}) + S(\rho^{B_2}) + \dots + S(\rho^{B_M}) - \max_{Z=B_1, B_2, \dots, B_M} \sum_x p_x S(\rho_x^Z). \quad (11)$$

Eq.(11) can be regarded as a generalization of the result of Ref.[11] to the case with multipartite senders and multipartite receivers.

In summary, we have proposed a universal Holevo-like upper bound on the locally accessible information for arbitrary multipartite ensembles. This bound allows us not only to prove the indistinguishability of some multipartite ensembles but also enables us to obtain the upper bound for the capacity of distributed dense coding with multipartite senders and multipartite receivers.

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